LETTER TO THE EDITOR

Mixing of ground states in vertex models

Jin Hong†‖, Seok-Jin Kang†¶, Tetsuji Miwa‡+ and Robert Weston§*

† Department of Mathematics, Seoul National University, Seoul 151-742, Korea
‡ Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan
§ Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, UK

Abstract. We consider the analogue of the six-vertex model constructed from alternating spin $n/2$ and spin $m/2$ lines, where $1 \leqslant n < m$. We identify the transfer matrix and the space on which it acts in terms of the representation theory of $U_q(\widehat{sl}_2)$. We diagonalize the transfer matrix and compute the S-matrix. We give a trace formula for local correlation functions. When $n = 1$, the one-point function of a spin $m/2$ local variable for the alternating lattice with a particular ground state is given as a linear combination of the one-point functions of the pure spin $m/2$ model with different ground states. The mixing ratios are calculated exactly and are expressed in terms of irreducible characters of $U_q(\widehat{sl}_2)$ and the deformed Virasoro algebra.

An algebraic description of the infinite-volume six-vertex model is given in [1, 2]. There are two key elements in this approach: to identify the half-infinite space $\cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ on which the corner transfer matrix acts, with a level-one irreducible highest-weight module $V(\Lambda_i)$ of $U_q(\widehat{sl}_2)$ ($i = 0, 1$) [3]; and to identify the half-transfer matrices on this space with components of the $U_q(\widehat{sl}_2)$ intertwiner $V(\Lambda_i) \rightarrow V(\Lambda_{1-i}) \otimes \mathbb{C}^2$ [4, 2].

The six-vertex model is associated with a lattice of intersecting lines, each carrying a spin $\frac{1}{2}$ representation of $U_q(\widehat{sl}_2)$. In this letter, we consider the analogous model on a lattice in which spin $n/2$ and spin $m/2$ ($1 \leqslant n < m$) lines alternate. Such models were constructed and analysed using the Bethe ansatz in [5–8]. The simplest example of the type of model we are considering is the lattice consisting of alternating spin $\frac{1}{2}$ and spin 1 lines, i.e. the case $n = 1$, $m = 2$ (see figure 2). The half-infinite space on which the corner transfer matrix of this model acts is $\cdots \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$. We shall develop an algebraic picture in which this space is identified with the tensor product $\mathcal{F}_{a,b} = V(\Lambda_a) \otimes V(\Lambda_b)$, $(a, b = 0, 1)$. There will now be two half-transfer matrices, associated with spin $\frac{1}{2}$ and spin 1 lines (see figures 3(*a*) and (*b*)), that we will identify with the $U_q(\widehat{sl}_2)$ intertwiners

$$\phi^A : V(\Lambda_a) \otimes V(\Lambda_b) \rightarrow V(\Lambda_a) \otimes V(\Lambda_{1-b}) \otimes \mathbb{C}^2$$

and

$$\phi^B : V(\Lambda_a) \otimes V(\Lambda_b) \rightarrow V(\Lambda_{1-a}) \otimes \mathbb{C}^2 \otimes V(\Lambda_b) \rightarrow V(\Lambda_{1-a}) \otimes V(\Lambda_{1-b}) \otimes \mathbb{C}^3$$

respectively. The full space $\cdots \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \cdots$ will then be identified with the level zero module $\mathcal{F}_{a,b} \otimes \mathcal{F}_{a,b}^*$.

‖ E-mail address: jhong@math.snu.ac.kr
¶ E-mail address: sjkang@math.snu.ac.kr
+ E-mail address: miwa@kurims.kyoto-u.ac.jp
* E-mail address: r.a.weston@ma.hw.ac.uk

We develop such a picture for the general case ($1 \leqslant n < m$), and construct the full transfer matrix in terms of the intertwiners ϕ^A and ϕ^B and their duals. We diagonalize the transfer matrix, and find a pseudoparticle spectrum consisting of only spin 0 and spin $\frac{1}{2}$ states. Such a spectrum was observed in the case $n = 1$, $m = 2$ in [7]. We compute the S-matrix for these excitations, and give a trace formula for local correlation functions of the theory.

Using our algebraic picture, we show that the one-point function of a spin $n/2$ variable is equal to that in the pure spin $n/2$ lattice when the ground states of the two different lattices are chosen correctly. However, we also show that the one-point function of a spin $m/2$ variable is a linear combination of the ones for the pure spin $m/2$ lattice corresponding to different ground states. We call this result a *mixing of ground states*.

After this brief introduction to some of our results, we will now define our alternating spin vertex model in some more detail. First of all, the Boltzmann weights are specified by the R-matrix $R^{(k,l)}(\zeta_1/\zeta_2) : V^{(k)}_{\zeta_1} \otimes V^{(l)}_{\zeta_2} \to V^{(k)}_{\zeta_1} \otimes V^{(l)}_{\zeta_2}$, where $V^{(k)}_\zeta \simeq \mathbb{C}^{k+1}$ is the $U_q(\widehat{sl}_2)$ principal evaluation module, with weight vectors $u^{(k)}_i$ ($i = 0, \ldots, k$), given in [9]. We fix the normalization such that $R^{(k,l)}(\zeta) = \bar{R}^{(k,l)}(\zeta)/\kappa^{(k,l)}(\zeta)$, where $\bar{R}^{(k,l)}(\zeta)(u^{(k)}_0 \otimes u^{(l)}_0) = (u^{(k)}_0 \otimes u^{(l)}_0)$, and

$$\kappa^{(k,l)}(\zeta) = \zeta^{\min(k,l)} \frac{(q^{2+k+l}\zeta^2; q^4)_\infty (q^{2+|k-l|}\zeta^{-2}; q^4)_\infty}{(q^{2+k+l}\zeta^{-2}; q^4)_\infty (q^{2+|k-l|}\zeta^2; q^4)_\infty}.$$

With this normalization, the partition function per unit site of our lattice model will be 1.

A lattice vertex associated with the intersection of a spin $k/2$ and spin $l/2$ line has six variables attached to it: spin variables $i, i' = (0, \ldots, k)$ and $j, j' = (0, \ldots, l)$, and spectral parameters $\zeta_1, \zeta_2 \in \mathbb{C}$. A Boltzmann weight $R^{(k,l)}(\zeta)^{i,j}_{i',j'}$ (with $\zeta = \zeta_1/\zeta_2$) is attached to the configuration of these variables shown in figure 1.

We choose to restrict our parameters to the region $-1 < q < 0$, $1 < \zeta < (-q)^{-1}$. With this restriction, the weights $R^{(k,l)}(\zeta)^{i,j}_{i',j'}$ with $k \leqslant l$ that are of the lowest order, and are larger than all other weights of higher order, are all those that obey the requirement $k \leqslant i + j \leqslant l$, $i' = k - i$, $j' = j + 2i - k$. For example, if $k = 1$ and $l = 2$, the largest Boltzmann weights will be $R^{(1,2)}(\zeta)^{0,1}_{1,0}$, $R^{(1,2)}(\zeta)^{0,2}_{1,1}$, $R^{(1,2)}(\zeta)^{1,0}_{0,1}$, and $R^{(1,2)}(\zeta)^{1,1}_{0,2}$.

Now consider the finite periodic lattice constructed by alternating $2M$ spin $n/2$ with $2M$ spin $m/2$ lines in both the horizontal and vertical directions. Such a lattice with $M = 2$ is shown in figure 2. In this figure, full lines represent spin $n/2$ lines, and dotted lines spin $m/2$ lines. The vertical lines have spectral parameter ζ, and horizontal lines have a spectral paramater equal to 1. We place two horizontal and two vertical spin $n/2$ lines next to each other at the centre of the lattice, and two vertical and two horizontal spin $m/2$ lines next to each other at the boundaries. We do this purely to simplify the discussion of the spaces on which the corner transfer matrices act. The order of the spin $n/2$ and spin $m/2$ lines can
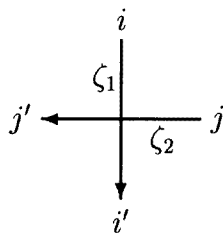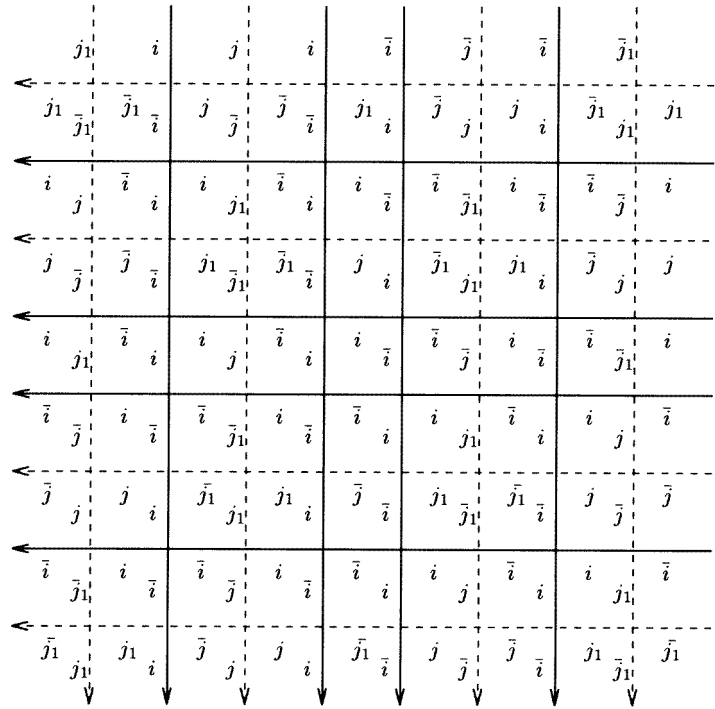

Figure 1.

Figure 2.

be switched to a uniform alternating pattern by using the Yang–Baxter equation.

There are $(n+1)(m-n+1)$ degenerate ground states of the vertex model associated with this lattice (by a ground state, we mean a configuration that contributes maximally to the partition function sum). The values of the spin variables in different ground states are shown in figure 2, in which we use the notation $j_1 = m+n-(2i+j)$, $\bar{i} = n-i$, $\bar{j} = m-j$, $\bar{j}_1 = m - j_1$. There is a different ground state corresponding to each choice of the pair of integers $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant m$ that obey the requirement $n \leqslant i + j \leqslant m$. We label such a ground state by the integers $0 \leqslant a \leqslant m - n$ and $0 \leqslant b \leqslant n$ defined by $a = m - i - j$ and $b = n - i$. We are interested in the infinite-volume limit of this lattice taken as follows: instead of having periodic boundary conditions, we fix the spins at the boundaries of the lattice to be equal to one of these ground states. We label the partition function by $Z_{a,b}^M$, and consider this lattice in the limit $M \to \infty$.

The infinite lattice so defined may be split into four quadrants. The associated corner transfer matrices are labelled $A_{\mathrm{NW}}(\zeta)$, $A_{\mathrm{SW}}(\zeta)$, $A_{\mathrm{SE}}(\zeta)$ and $A_{\mathrm{NE}}(\zeta)$. A *path* is defined as $|p\rangle = p(s), (s > 0)$, where $p(s)$ is the value of the spin variable on the sth line (our convention is that s increases from east to west, and from south to north). $A_{\mathrm{NW}}(\zeta)$ acts on the path space $P_{a,b}$ defined by

$$P_{a,b} = \{|p\rangle \mid p(s) = \bar{p}(s; a, b), s \gg 0\}$$

where

$$\bar{p}(s; a, b) = \begin{cases} n - b & \text{if } s \text{ is odd} \\ m - n - a + b & \text{if } s \equiv 2 \bmod 4 \\ a + b & \text{if } s \equiv 4 \bmod 4. \end{cases}$$

We have $A_{\text{NW}}(\zeta) : P_{a,b} \to P_{a,b}$, $A_{\text{SW}}(\zeta) : P_{a,b} \to P_{m-n-a,n-b}$, $A_{\text{SE}}(\zeta) : P_{m-n-a,n-b} \to P_{m-n-a,n-b}$, and $A_{\text{NE}}(\zeta) : P_{m-n-a,n-b} \to P_{a,b}$. In fact, using the crossing symmetry of the R-matrix, we can write

$$A_{\text{SW}}(\zeta) = S A_{\text{NW}}(-q^{-1}\zeta^{-1}) \qquad A_{\text{SE}}(\zeta) = S A_{\text{NW}}(\zeta) S \qquad A_{\text{NE}}(\zeta) = A_{\text{NW}}(-q^{-1}\zeta^{-1}) S.$$

Here $S : P_{a,b} \to P_{m-n-a,n-b}$ is the operator defined by

$$p(s) \longmapsto \begin{cases} n - p(s) & \text{if } s \text{ is odd} \\ m - p(s) & \text{if } s \text{ is even.} \end{cases}$$

Baxter's argument [10] then implies that we have $A_{\text{NW}}(\zeta) = (\text{constant})\, \zeta^{-H_{\text{CTM}}}$, where H_{CTM} is independent of ζ and has a non-negative integer spectrum. The partition function in the infinite lattice is

$$Z_{a,b} = \text{Tr}_{P_{a,b}}(A_{\text{NE}}(\zeta) A_{\text{SE}}(\zeta) A_{\text{SW}}(\zeta) A_{\text{NW}}(\zeta)) \propto \text{Tr}_{P_{a,b}}(q^{2H_{\text{CTM}}}).$$

The next step is to understand the path space $P_{a,b}$ in terms of the representation theory of $U_q(\widehat{sl}_2)$. The $U_q(\widehat{sl}_2)$ modules relevant to our discussion are the level $\ell > 0$ irreducible highest-weight modules $V(\lambda_r^{(\ell)})$. These are generated by the level ℓ highest-weight vector $v_{\lambda_r^{(\ell)}}$ ($\lambda_r^{(\ell)} = r\Lambda_1 + (\ell - r)\Lambda_0, 0 \leqslant r \leqslant \ell$), which obeys $e_1 v_{\lambda_r^{(\ell)}} = e_0 v_{\lambda_r^{(\ell)}} = f_1^{r+1} v_{\lambda_r^{(\ell)}} = f_0^{\ell-r+1} v_{\lambda_r^{(\ell)}} = 0$. We use a grading element D which acts on $V(\lambda_r^{(\ell)})$ as $D(f_{i_1} f_{i_2} \ldots f_{i_N} v_{\lambda_r^{(\ell)}}) = N(f_{i_1} f_{i_2} \ldots f_{i_N} v_{\lambda_r^{(\ell)}})$. Labelling $\mathcal{H}_{a,b} = V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)})$, we conjecture that we can identify $P_{a,b} = \mathcal{H}_{a,b}$ and $H_{\text{CTM}} = D$. We have a proof of this identification in the $q \to 0$ limit, but omit it here for lack of space.

Now we come to the identification of half-transfer matrices. There are two half-transfer matrices for our lattice, associated with the insertion of half-infinite spin $n/2$ and spin $m/2$ lines respectively (see figures 3(a) and (b)).

We shall identify these with certain components of the following intertwiners:

$$\phi^A(\zeta) : V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \to V(\lambda_a^{(m-n)}) \otimes V(\sigma(\lambda_b^{(n)})) \otimes V_\zeta^{(n)}$$

$$\phi^B(\zeta) : V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \to V(\sigma(\lambda_a^{(m-n)})) \otimes V_\zeta^{(m-n)} \otimes V(\lambda_b^{(n)})$$

$$\to V(\sigma(\lambda_a^{(m-n)})) \otimes V(\sigma(\lambda_b^{(n)})) \otimes V_\zeta^{(m)}$$
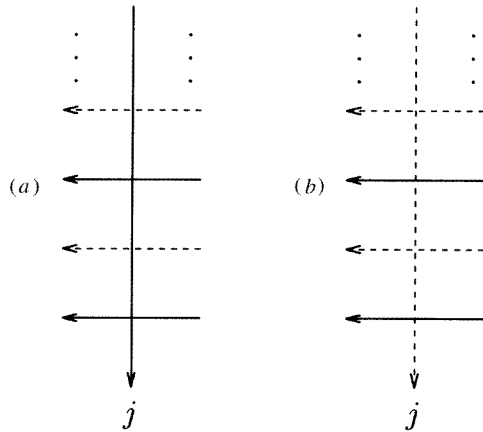

(a) (b)

$\hat{j}$ $\hat{j}$

Figure 3.

where σ exchanges the fundamental weights $\Lambda_0 \leftrightarrow \Lambda_1$. These intertwiners are in turn defined in terms of the more elementary intertwiner

$$\Phi^{(k,k+\ell)}(\zeta) : V_\zeta^{(k)} \otimes V(\lambda_r^{(\ell)}) \to V(\sigma(\lambda_r^{(\ell)})) \otimes V_\zeta^{(k+\ell)} \qquad k, \ell \geqslant 1. \qquad (1)$$

We define $\Phi^{(0,\ell)}(\zeta)$ as the intertwiner $V(\lambda_r^{(\ell)}) \to V(\sigma(\lambda_r^{(\ell)})) \otimes V_\zeta^{(\ell)}$. We define the components of $\Phi^{(k,k+\ell)}(\zeta)$ by

$$\Phi^{(k,k+\ell)}(u_i^{(k)} \otimes v) = \sum_{j=0}^{k+\ell} \Phi_{i,j}^{(k,k+\ell)}(v) \otimes u_j^{(k+\ell)} \qquad v \in V(\lambda_r^{(\ell)}).$$

The normalization of $\Phi^{(k,k+\ell)}(\zeta)$ is fixed by the requirement

$$\langle \sigma(\lambda_r^{(\ell)}) | \Phi_{0,\ell-r}^{(k,k+\ell)}(\zeta) | \lambda_r^{(\ell)} \rangle = 1.$$

The intertwiner $\Phi^{(k,k+\ell)}(\zeta)$ is the generalization to level $\ell \geqslant 1$ of the intertwiner introduced by Nakayashiki [11]; its crystal limit was considered in [12, 13] (see also [9]). $\Phi^{(k,k+\ell)}(\zeta)$ has the following properties:

$$\xi^{-D} \Phi_{i,j}^{(k,k+\ell)}(\zeta) \xi^D = \Phi_{i,j}^{(k,k+\ell)}(\zeta/\xi)$$

$$g^{(k,k+\ell)} \sum_{j=0}^{k+\ell} \Phi_{i,j}^{(k,k+\ell)}(-q^{-1}\zeta) \Phi_{k-i',k+\ell-j}^{(k,k+\ell)}(\zeta) = \delta_{i,i'} \qquad (2)$$

where $g^{(k,k+\ell)} = (q^{2k+2}; q^4)_\infty / (q^{2(k+\ell)+2}; q^4)_\infty$.

If $v \otimes v' \in V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)})$, then we define $\phi^A(\zeta)$ and $\phi^B(\zeta)$ by

$$\phi^A(v \otimes v') = \sum_{j=0}^{n} \phi_j^A(v \otimes v') \otimes u_j^{(n)} \qquad \phi_j^A(v \otimes v') = v \otimes \Phi_j^{(0,n)} v'$$

$$\phi^B(v \otimes v') = \sum_{j=0}^{m} \phi_j^B(v \otimes v') \otimes u_j^{(m)} \qquad \phi_j^B(v \otimes v') = \sum_{j'=0}^{m-n} \Phi_{j'}^{(0,m-n)} v \otimes \Phi_{j',j}^{(m-n,m)} v'. \qquad (3)$$

Here, for clarity, we have suppressed the ζ-dependence of all our intertwiners. It is the components $\phi_j^A(\zeta)$ ($j = 0, \ldots, n$), and $\phi_j^B(\zeta)$ ($j = 0, \ldots, m$), that we identify with the lattice insertions shown in figures 3(a) and (b) respectively.

Now define $\mathcal{H} = \sum_{a,b} \mathcal{H}_{a,b}$ with $0 \leqslant a \leqslant m - n$ and $0 \leqslant b \leqslant n$, and $\mathcal{F} = \mathcal{H} \otimes \mathcal{H}^*$. We shall identify $\mathcal{F}$ with the space on which our full transfer matrix acts. Note that we can view $\mathcal{F}$ as a linear map on $\mathcal{H}$ via the canonical identification $\mathcal{H} \otimes \mathcal{H}^* \simeq \mathrm{End}\,(\mathcal{H})$. The dual space $\mathcal{F}^*$ is defined via $\langle f | g \rangle = \mathrm{Tr}_{\mathcal{H}}(f \circ g)$. The transfer matrix itself is defined as $\mathcal{T}(\zeta) = T^B(\zeta) \circ T^A(\zeta)$, where $T^A(\zeta)$ and $T^B(\zeta)$ represent the insertion of double-infinite vertical spin $n/2$ and spin $m/2$ lines respectively. They are given by

$$T^A(\zeta) = \sum_{j=0}^{n} T_j^A(\zeta) \qquad T_j^A(\zeta) = g^{(0,n)} \phi_j^A(\zeta) \otimes \phi_{n-j}^A(\zeta)^t$$

$$T^B(\zeta) = \sum_{j=0}^{m} T_j^B(\zeta) \qquad T_j^B(\zeta) = g^{(0,m)} \phi_j^B(\zeta) \otimes \phi_{m-j}^B(\zeta)^t \qquad (4)$$

where t denotes the transpose. The motivation for the identification of the full space and for the definition of the transfer matrices is essentially the same as that described in [2].

A vacuum state is a maximal eigenstate of $\mathcal{T}(\zeta)$. There are $(n+1)(m-n+1)$ degenerate vacua, labelled as $| \mathrm{vac} \rangle_{a,b} \in \mathcal{H}_{a,b} \otimes \mathcal{H}_{a,b}^*$, with $0 \leqslant a \leqslant m - n, 0 \leqslant b \leqslant n$. We conjecture

that, expressed as an element of End $(\mathcal{H})$, the vacuum $|\,\text{vac}\rangle_{a,b}$ and its dual $_{a,b}\langle\text{vac}|$ are given by

$$|\,\text{vac}\rangle_{a,b} = (\chi_a^{(m-n)}\chi_b^{(n)})^{-\frac{1}{2}}(-q)^D\pi_{a,b} \qquad _{a,b}\langle\text{vac}| = (\chi_a^{(m-n)}\chi_b^{(n)})^{-\frac{1}{2}}\pi_{a,b}(-q)^D.$$

Here, $\chi_r^{(\ell)} = \text{Tr}_{V(\lambda_r^{(\ell)})}(q^{2D})$, and $\pi_{a,b}$ is the projector $\mathcal{H} \to \mathcal{H}_{a,b}$. Using the definitions (3) and (4), and properties (2), it is then simple to show that $\mathcal{T}(\zeta)|\,\text{vac}\rangle_{a,b} = |\,\text{vac}\rangle_{m-n-a,b}$.

In order to consider other eigenstates of $\mathcal{T}(\zeta)$ apart from the vacua, we must first introduce some more intertwiners. On the level ℓ irreducible highest-weight modules, we have

$$\Psi^{*s}(\xi) : V_\xi^{(1)} \otimes V(\lambda_r^{(\ell)}) \to V(\lambda_{r+s}^{(\ell)})$$

$$\Phi^s(\xi) : V(\lambda_r^{(\ell)}) \to V(\lambda_{r+s}^{(\ell)}) \otimes V_\xi^{(1)}$$

where $s = \pm 1$ (or $\pm$ for short). For $v \in V(\lambda_r^{(\ell)})$, we define components by

$$\Psi^{*\pm}(u_\varepsilon^{(1)} \otimes v) = \Psi_\varepsilon^{*\pm}(v) \qquad (\varepsilon = 0, 1)$$

$$\Phi^\pm(v) = \sum_{\varepsilon=0}^1 \Phi_\varepsilon^\pm(v) \otimes u_\varepsilon^{(1)}$$

where we again suppress the ξ-dependence. We fix the normalization of these intertwiners to be

$$\langle\lambda_{r-1}^l|\Psi_1^{*-}(\xi)|\lambda_r^l\rangle = 1 \qquad \langle\lambda_{r+1}^l|\Psi_0^{*+}(\xi)|\lambda_r^l\rangle = 1$$

$$\langle\lambda_{r-1}^l|\Phi_0^-(\xi)|\lambda_r^l\rangle = 1 \qquad \langle\lambda_{r+1}^l|\Phi_1^+(\xi)|\lambda_r^l\rangle = 1.$$

They have the following commutation relations with the intertwiner $\Phi^{(k,l)}(\zeta)$ defined in equation (1):

$$\Phi_{i,j}^{(k,l)}(\zeta)\Psi_\varepsilon^{*\pm}(\xi) = \sum_{i',\varepsilon'} \Psi^{*\mp}(\xi)_{\varepsilon'}\Phi_{i',j}^{(k,l)}(\zeta)R^{(k,1)}(\zeta/\xi)_{i',\varepsilon'}^{i,\varepsilon}$$

$$\Phi_j^{(0,l)}(\zeta)\Psi_\varepsilon^{*\pm}(\xi) = \Psi^{*\mp}(\xi)_\varepsilon\Phi_j^{(0,l)}(\zeta)\tau(\zeta/\xi) \tag{5}$$

$$\Phi_j^{(0,l)}(\zeta)\Phi_\varepsilon^\pm(\xi) = \sum_{\varepsilon',j'} R^{(1,l)}(\xi/\zeta)_{\varepsilon,j}^{\varepsilon',j'}\Phi_{\varepsilon'}^\mp(\xi)\Phi_{j'}^{(0,l)}(\zeta)$$

where

$$\tau(\zeta) = \zeta^{-1}\frac{(q\zeta^2;q^4)_\infty(q^3\zeta^{-2};q^4)_\infty}{(q\zeta^{-2};q^4)_\infty(q^3\zeta^2;q^4)_\infty}.$$

We are now in a position to define our intertwiners on the tensor product space, and to complete the diagonalization of our transfer matrix $\mathcal{T}(\zeta)$. The intertwiners we require are

$$\psi^{(\frac{1}{2})s}(\xi) : V_\xi^{(1)} \otimes V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \to V(\lambda_{a+s}^{(m-n)}) \otimes V(\lambda_b^{(n)})$$

and

$$\psi^{(0)s,\tilde{s}}(\xi) : V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \to V(\lambda_{a+s}^{(m-n)}) \otimes V_\xi^{(1)} \otimes V(\lambda_b^{(n)}) \to V(\lambda_{a+s}^{(m-n)}) \otimes V(\lambda_{b+\tilde{s}}^{(n)})$$

where again $s, \tilde{s} = \pm 1$. On the element $v \otimes v' \in V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)})$, they are given in terms of the above intertwiners by

$$\psi^{(\frac{1}{2})s}(\xi)(u_\varepsilon^{(1)} \otimes v \otimes v') = \psi_\varepsilon^{(\frac{1}{2})s}(\xi)(v \otimes v') = (\Psi_\varepsilon^{*s}(\xi)v) \otimes v'$$

$$\psi^{(0)s,\tilde{s}}(\xi)(v \otimes v') = \sum_{\varepsilon=0}^1 (\Phi_\varepsilon^s(\xi)v) \otimes (\Psi_\varepsilon^{*\tilde{s}}(\xi)v').$$

Using the commutation relations (5) and the unitarity property of the R-matrix, it is then straightforward to show that

$$T^A(\zeta)\psi_\varepsilon^{(\frac{1}{2})s}(\xi)|\,\text{vac}\rangle_{a,b} = \psi_\varepsilon^{(\frac{1}{2})s}(\xi)|\,\text{vac}\rangle_{a,n-b}$$

$$T^B(\zeta)\psi_\varepsilon^{(\frac{1}{2})s}(\xi)|\,\text{vac}\rangle_{a,b} = \tau(\zeta/\xi)\psi_\varepsilon^{(\frac{1}{2})-s}(\xi)|\,\text{vac}\rangle_{m-n-a,n-b}$$

$$T^A(\zeta)\psi^{(0)\,s,\tilde{s}}(\xi)|\,\text{vac}\rangle_{a,b} = \tau(\zeta/\xi)\psi^{(0)\,s,-\tilde{s}}(\xi)|\,\text{vac}\rangle_{a,n-b}$$

$$T^B(\zeta)\psi^{(0)\,s,\tilde{s}}(\xi)|\,\text{vac}\rangle_{a,b} = \psi^{(0)\,-s,-\tilde{s}}(\xi)|\,\text{vac}\rangle_{m-n-a,n-b}$$

and hence that

$$\mathcal{T}(\zeta)\psi_\varepsilon^{(\frac{1}{2})s}(\xi)|\,\text{vac}\rangle_{a,b} = \tau(\zeta/\xi)\psi_\varepsilon^{(\frac{1}{2})-s}(\xi)|\,\text{vac}\rangle_{m-n-a,b}$$

$$\mathcal{T}(\zeta)\psi^{(0)\,s,\tilde{s}}(\xi)|\,\text{vac}\rangle_{a,b} = \tau(\zeta/\xi)\psi^{(0)\,-s,\tilde{s}}(\xi)|\,\text{vac}\rangle_{m-n-a,b}\,.$$

The two eigenstates $\psi_\varepsilon^{(\frac{1}{2})s}(\xi)|\,\text{vac}\rangle_{a,b}$ and $\psi^{(0)\,s,\tilde{s}}(\xi)|\,\text{vac}\rangle_{a,b}$ are spin $\frac{1}{2}$ and spin 0 states respectively. The spin $\frac{1}{2}$ excitation has a $T^A(\zeta)$ eigenvalue of 1 and a $T^B(\zeta)$ eigenvalue of $\tau(\zeta/\xi)$. The spin 0 excitation has a $T^A(\zeta)$ eigenvalue of $\tau(\zeta/\xi)$ and a $T^B(\zeta)$ eigenvalue of 1. This is the same pattern for the spectrum as that observed in the Bethe ansatz calculations of [7] for the $n = 1, m = 2$ case. Multiparticle states arise as compositions of $\psi^{(\frac{1}{2})s_i}(\xi_i)_{\varepsilon_i}$ and $\psi^{(0)s_i',\tilde{s}_i'}(\xi_{i'})$ acting on $|\,\text{vac}\rangle_{a,b}$. The eigenvalue of $\mathcal{T}(\zeta)$ is then simply the product of all the $\tau(\zeta/\xi_i)$ and $\tau(\zeta/\xi_{i'})$ factors.

The S-matrix of the excitations in our model is given by the following relations:

$$\psi_{\varepsilon_1}^{(\frac{1}{2})s_1}(\xi_1)\psi_{\varepsilon_2}^{(\frac{1}{2})s_2}(\xi_2) = \sum_{\varepsilon_1',\varepsilon_2',s_1',s_2'} \psi_{\varepsilon_2'}^{(\frac{1}{2})s_2'}(\xi_2)\psi_{\varepsilon_1'}^{(\frac{1}{2})s_1'}(\xi_1)\bar{R}^{(1,1)}(\xi)_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1',\varepsilon_2'}W^{II}\left(\begin{array}{cc}\lambda & \mu \\ \mu' & \nu\end{array}\Big|\xi\right)$$

$$\psi^{(0)\,s_1,\tilde{s}_1}(\xi_1)\psi^{(0)\,s_2,\tilde{s}_2}(\xi_2)$$
$$= \sum_{s_1',s_2',\tilde{s}_1',\tilde{s}_2'} \psi^{(0)\,s_2',\tilde{s}_2'}(\xi_2)\psi^{(0)\,s_1',\tilde{s}_1'}(\xi_1)W^{I}\left(\begin{array}{cc}\lambda & \mu \\ \mu' & \nu\end{array}\Big|\xi\right)W^{II}\left(\begin{array}{cc}\tilde{\lambda} & \tilde{\mu} \\ \tilde{\mu}' & \tilde{\nu}\end{array}\Big|\xi\right)$$

$$\psi^{(0)s_1,\tilde{s}_1}(\xi_1)\psi_{\varepsilon_2}^{(\frac{1}{2})s_2}(\xi_2) = \sum_{s_1',s_2'} \psi_{\varepsilon_2}^{(\frac{1}{2})s_2'}(\xi_2)\psi^{(0)s_1',\tilde{s}_1}(\xi_1)W^{*}\left(\begin{array}{cc}\lambda & \mu \\ \mu' & \nu\end{array}\Big|\xi\right)$$

where $\xi = \xi_1/\xi_2$. We use the notation that λ, μ, μ', ν are dominant integral weights of level $m - n$, with λ, ν fixed and $\mu = \lambda + s_2\bar{\rho}$, $\mu' = \lambda + s_1'\bar{\rho}$, $\nu = \lambda + (s_1 + s_2)\bar{\rho}$ (where $\bar{\rho} = \Lambda_1 - \Lambda_0$). The s_1', s_2' sum is over all $s_1', s_2' = \pm 1$ such that $\nu = \lambda + (s_1' + s_2')\bar{\rho}$. The notation and summations for the tilded quantities are the same, except that the weights $\tilde{\lambda}, \tilde{\mu}, \tilde{\mu}', \tilde{\nu}$ are of level n. The remaining quantities are given for level ℓ weights by

$$W^{I}\left(\begin{array}{cc}\lambda_r^{(\ell)} & \lambda_s^{(\ell)} \\ \lambda_u^{(\ell)} & \lambda_t^{(\ell)}\end{array}\Big|\xi\right) = \frac{X(p^2\xi^{-2})}{X(p^2\xi^2)}\overline{W}_\ell^1\left(\begin{array}{cc}\lambda_r^{(\ell)} & \lambda_s^{(\ell)} \\ \lambda_u^{(\ell)} & \lambda_t^{(\ell)}\end{array}\Big|\xi^2\right)\xi^{\delta_{t,s+1}-\delta_{r,u-1}}$$

$$W^{II}\left(\begin{array}{cc}\lambda_r^{(\ell)} & \lambda_s^{(\ell)} \\ \lambda_u^{(\ell)} & \lambda_t^{(\ell)}\end{array}\Big|\xi\right) = \frac{X(\xi^{-2})}{X(\xi^2)}\overline{W}_\ell^1\left(\begin{array}{cc}\lambda_r^{(\ell)} & \lambda_s^{(\ell)} \\ \lambda_u^{(\ell)} & \lambda_t^{(\ell)}\end{array}\Big|\xi^2\right)\xi^{\delta_{t,s+1}-\delta_{r,u-1}}$$

$$W^{*}\left(\begin{array}{cc}\lambda_r^{(\ell)} & \lambda_s^{(\ell)} \\ \lambda_u^{(\ell)} & \lambda_t^{(\ell)}\end{array}\Big|\xi\right) = \frac{X(p\xi^{-2})}{X(p\xi^2)}\overline{W}_\ell^1\left(\begin{array}{cc}\lambda_r^{(\ell)} & \lambda_s^{(\ell)} \\ \lambda_u^{(\ell)} & \lambda_t^{(\ell)}\end{array}\Big|p^{-1}\xi^2\right)(-\xi q^{-(1+r)})^{\delta_{t,s+1}-\delta_{r,u-1}}q^{\delta_{r,t}}$$

where

$$X(z) = \frac{(z;p^2,q^4)_\infty(q^4z;p^2,q^4)_\infty}{(q^2z;p^2,q^4)_\infty^2} \qquad p = q^{\ell+2}$$

and $\overline{W}_\ell^1$ is the RSOS Boltzmann weight in the notation of equation (B.2) in [14].

When $n = 1$, $m = 2$ (the alternating spin $\frac{1}{2}$, spin 1 model), the above commutation relations become

$$\psi_{\varepsilon_1}^{(\frac{1}{2})}(\xi_1)\psi_{\varepsilon_2}^{(\frac{1}{2})}(\xi_2) = -\sum_{\varepsilon_1',\varepsilon_2'} \psi_{\varepsilon_2'}^{(\frac{1}{2})}(\xi_2)\psi_{\varepsilon_1'}^{(\frac{1}{2})}(\xi_1) R^{(1,1)}(\xi)_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1',\varepsilon_2'}$$

$$\psi^{(0)}(\xi_1)\psi^{(0)}(\xi_2) = -\psi^{(0)}(\xi_2)\psi^{(0)}(\xi_1)$$

$$\psi^{(0)}(\xi_1)\psi_{\varepsilon_2}^{(\frac{1}{2})}(\xi_2) = \tau(\xi)\psi_{\varepsilon_2}^{(\frac{1}{2})}(\xi_2)\psi^{(0)}(\xi_1).$$

Here, we have suppressed the appearance of s_i and $\tilde{s}_i$, as in this case they are determined solely by the space on which the operators act. These relations are consistent with the results of [7].

We have now reached the point where we can discuss correlation functions and the mixing of ground states. A correlation function of the infinite-volume vertex model we are discussing is by definition a ratio $Z_{a,b}^{\{i_N,...,i_1\}}/Z_{a,b}$. Here, $Z_{a,b}$ is the partition function described earlier, and $Z_{a,b}^{\{i_N,...,i_1\}}$ is a similarly weighted sum over a restricted set of spin configurations. The restriction is to include only those configurations for which the spin variables on some specified N edges have the fixed values $i_N, \ldots, i_1$. Suppose that N is odd, and that the N edges are vertical ones located successively in one row. Then, our correlation function will be given by the expression

$$
\begin{aligned}
F_{a,b}^{\{i_N,...,i_1\}} =_{a,b} &\langle \text{vac}| T_{i_N}^B(1) T_{i_{N-1}}^A(1) T_{i_{N-2}}^B(1) \cdots T_{i_2}^A(1) T_{i_1}^B(1) | \text{vac}\rangle_{a',n-b} \\
= &(g^{(0,m)})^{\frac{N+1}{2}} (g^{(0,n)})^{\frac{N-1}{2}} (\chi_a^{(m-n)}\chi_b^{(n)})^{-1} \\
&\times \text{Tr}_{\mathcal{H}_{a,b}}(q^{2D}\phi_{i_N}^B(-q^{-1})\phi_{i_{N-1}}^A(-q^{-1})\phi_{i_{N-2}}^B(-q^{-1}) \cdots \phi_{i_2}^A(-q^{-1})\phi_{i_1}^B(-q^{-1}) \\
&\times \phi_{m-i_1}^B(1)\phi_{n-i_2}^A(1) \cdots \phi_{m-i_{N-2}}^B(1)\phi_{n-i_{N-1}}^A(1)\phi_{m-i_N}^B(1))
\end{aligned}
\tag{6}
$$

where

$$
a' = \begin{cases} m - n - a & \text{for } N = 1 \bmod 4 \\ a & \text{for } N = 3 \bmod 4. \end{cases}
$$

(See [2] for a detailed explanation of this expression for the spin $\frac{1}{2}$ case; the generalization to our case is straightforward.)

Now let us consider the case when $n = N = 1$. Our formula (6) then gives us the one-point function which corresponds to a restricted sum in which the local variable on the central edge of the central vertical spin $m/2$ line is fixed (in an alternating spin $\frac{1}{2}$, spin $m/2$ lattice). Suppose instead that we had started out with an infinite lattice consisting of just spin $m/2$ lines. The algebraic picture of such pure models is considered in [15]. The question arises as to whether, and how, the one-point function in our alternating spin lattice might be related to the one-point in the pure lattice. We can supply a straightforward answer to this question using the algebraic analysis we have described.

The argument is as follows. When $n = 1$, $m > 1$, our half-space is $\mathcal{H}_{a,b} = V(\lambda_a^{(m-1)}) \otimes V(\Lambda_b)$ where $0 \leqslant a \leqslant m - 1$, $0 \leqslant b \leqslant 1$. In this case, we have the irreducible decomposition

$$V(\lambda_a^{(m-1)}) \otimes V(\Lambda_b) = \oplus_{c=0}^m V(\lambda_c^{(m)}) \otimes \Omega_c^{a,b} \tag{7}$$

where the sum runs over the level m weights $\lambda_c^{(m)}$, and $\Omega_c^{a,b}$ is the space of highest-weight states in $V(\lambda_a^{(m-n)}) \otimes V(\Lambda_b)$. In a recent paper [16], Jimbo and Shiraishi constructed the action of the deformed Virasoro algebra on $\Omega_c^{a,b}$ by making use of the operator

$$\psi^{(0)}(\zeta) : V(\lambda_a^{(m-1)}) \otimes V(\Lambda_b) \to V(\lambda_{a'}^{(m-1)}) \otimes V_\zeta^{(1)} \otimes V(\Lambda_b) \to V(\lambda_{a'}^{(m-1)}) \otimes V(\Lambda_{1-b})$$

which we have defined above (again we suppress the s and $\tilde{s}$ indices of $\psi^{(0)}(\zeta)$). One can prove that

$$\phi^B(\zeta)\psi^{(0)}(\xi) = \psi^{(0)}(\xi)\phi^B(\zeta).$$

Hence, $\phi^B(\zeta)$ commutes with the action of the deformed Virasoro algebra; it acts only on the space $V(\lambda_c^{(m)})$ in (7).

Now consider the consequences for the one-point function discussed above, (given by (6) when $N = n = 1$). The trace simply splits up into the trace over the different level m highest-weight modules in (7). Specifically, we have

$$F_{a,b}^{\{k\}} = \frac{\sum_{c=0}^{m} F_c^{\{k\}} \chi_c^{(m)} \chi_{\Omega_c^{a,b}}}{\sum_{c=0}^{m} \chi_c^{(m)} \chi_{\Omega_c^{a,b}}} \tag{8}$$

where $F_c^{\{k\}}(\zeta)$, with $(k = 0, \ldots m)$, is the corresponding one-point function for the pure spin $m/2$ lattice with the ground state labelled by c, and $\chi_{\Omega_c^{a,b}} = \mathrm{Tr}_{\Omega_c^{a,b}}(q^{2D})$. Thus, the one-point function for the alternating lattice with a particular ground state is given as a linear combination of the one-point functions associated with different ground states in the pure lattice; there is a mixing of ground states.

Let us briefly discuss the relation between the above result and Baxter's argument in [17]. Consider the simplest case; in which the lattice consists of spin $\frac{1}{2}$ and spin 1 lines. The idea of Baxter is to change the arrangement of the lines without changing the configuration sums. This is possible because of the Yang–Baxter equation for the Boltzmann weights. A configuration sum consists of the contributions of the chosen ground state and other states which are modified from it at finitely many edges. Namely, it is a sum over diagrams consisting of those edges on which the values of the local variables differ from the ground state values. Usually, in such an expansion, contributions of larger diagrams are of higher order in q. Suppose this were true for the alternating lattice; in order to compute the one-point function of a spin 1 local variable on a given edge, we could move the spin $\frac{1}{2}$ lines far away from this edge. Then, small diagrams of low order would only be associated with the spin 1 lines near the chosen edge. This would imply that the one-point functions of the alternating lattice should be the same as the one-point functions of the pure spin 1 lattice. This is in contradiction with our results from the representation theory. However, the above assumption about the size and the order of diagrams is not correct in the alternating lattice. Consider a rectangular region consisting of spin 1 edges that is bordered by four spin $\frac{1}{2}$ lines. Take the ground state configuration on this lattice corresponding to $(a, b) = (0, 0)$. This ground state consists of alternating values 0 and 2 on the spin 1 edges in this region, and 0 and 1 on the spin $\frac{1}{2}$ edge on its border. If we switch the configuration of these variables (i.e. $0 \leftrightarrow 2$ and $0 \leftrightarrow 1$), the order of the contribution is q^2 no matter how large the region is. This explains the origin of mixing in the small q expansion.

In fact we have performed such a small q expansion for our alternating lattice, and calculated up to order q^4 all the different one-point functions corresponding to the left-hand side of (8) when $m = 2$. The level two one-point functions on the right-hand side of the equation were calculated by Idzumi [18]. To this order, the results agree with the mixing formula and disagree with the pure one-point function.

Finally, suppose we wish to calculate a correlation function such as

$$_{a,b}\langle \mathrm{vac}|T_k^A(\zeta)|\mathrm{vac}\rangle_{a,n-b} = g^{(0,n)}(\chi_a^{(m-n)}\chi_b^{(n)})^{-1}\mathrm{Tr}_{\mathcal{H}_{a,b}}(q^{2D}\phi_k^A(-q^{-1}\zeta)\phi_{n-k}^A(\zeta)) \tag{9}$$

that only involves the insertion of spin $n/2$ lines into our alternating lattice. From its definition (3), ϕ^A acts just on the right-hand side of the tensor product $\mathcal{H}_{a,b} =$

$V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)})$, and so (9) is equal to

$$g^{(0,n)}(\chi_b^{(n)})^{-1} \operatorname{Tr}_{V(\lambda_b^{(n)})}(q^{2D}\Phi_k^{(0,n)}(-q^{-1}\zeta)\Phi_{n-k}^{(0,n)}(\zeta)).$$

This is the one-point function associated with the pure spin $n/2$ model with a single ground state. There is no mixing of ground states when we calculate correlation functions involving only insertions of spin $n/2$ lines into our alternating lattice. The physical argument that was previously wrong is now right. If we move spin $m/2$ lines far away from our central fixed spin $n/2$ lines, then diagrams that are of low order in q in the small q expansion are indeed small in size and restricted to the pure spin $n/2$ region.

In summary, we have constructed an algebraic picture of infinite-volume alternating spin $n/2$, spin $m/2$ $(1 \leqslant n < m)$ vertex models. We have diagonalized the transfer matrix. The eigenstates are made up from spin $\frac{1}{2}$ and spin 0 particle-like excitations. We have computed the two-particle S-matrix elements for these excitations. A trace formula is given for local correlation functions. Specializing to $1 = n < m$, we have expressed the one-point function associated with a spin $m/2$ line as a linear combination of the corresponding one-point functions in the pure spin $m/2$ model with different ground states. For the general alternating model $(1 \leqslant n < m)$, we have shown that the one-point function associated with a spin $n/2$ line is equal to the corresponding one-point function in the pure spin $n/2$ model, without mixing of ground states.

In this letter, we have presented in brief the main results of our analysis of alternating spin vertex models. A more detailed explanation will be published elsewhere.

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